Interacting criteria in MultiCriteria Decision Making

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Outline

1. Multiattribute utility theory (MAUT)
2. The Choquet integral and MLE models
3. GAI models
4. Interaction between criteria

Notation, basic notions

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- Example: the *additive utility model*

$$U(x) = \sum_{i \in N} u_i(x_i)$$

Preferential independence

- $A \subset N$ is *preferentially independent* of its complement $N \setminus A$ if for every $x, y, z, z' \in X$

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- Does not always hold! Example: evaluation of students. The following preference reversal is not unlikely:

	Mathematics	Physics	Language skills
Student A	40	90	60
Student B	40	60	90
Student C	80	90	60
Student D	80	60	90

$A \succ B$ and $C \prec D$

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- The additive utility model implies preferential independence

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- Under weak independence, one can define on each attribute X_i a preference relation $\succcurlyeq_i$:

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  for some z_{-i}
- Under weak independence and order density, $\succcurlyeq$ can be represented by the *decomposable model*

$$U(x) = F(u_1(x_1), \dots, u_n(x_n))$$

  with F a strictly increasing function, and the u_i's are utility functions representing $\succcurlyeq_i$.

Weak independence

Although standard in decision models, the condition does not always hold: see the menu example.

$$\text{(meat,red wine)} \succ \text{(meat, white wine)}$$
$$\text{(fish,red wine)} \prec \text{(fish, white wine)}$$

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- A generalization of the weighted arithmetic mean is given by the Choquet integral model and the MLE model: they are based on a generalized set of weights, called a capacity.

Interacting criteria

Let a, b, c be three alternatives evaluated on 2 criteria as follows:

$$\begin{aligned} u_1(a_1) = 0.4, \quad u_1(b_1) = 0, \quad u_1(c_1) = 1 \\ u_2(a_2) = 0.4, \quad u_2(b_2) = 1, \quad u_2(c_2) = 0, \end{aligned}$$

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- Solution: put a weight w_{12} on the *group of criteria* 1 and 2, expressing the fact that it is important that both criteria are satisfied, not only one.

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- More precisely, letting $\mathbf{1}_i, \mathbf{0}_i$ be particular elements of X_i (e.g., satisfactory, unsatisfactory), and fixing $u_i(\mathbf{1}_i) = 1$, $u_i(\mathbf{0}_i) = 0$,
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- It follows that F can be seen as an extension of v on $[0, 1]^n$

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- and m_A is the *Möbius transform of the set function v corresponding to f*, given by

$$m_A = m^v(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} v(B)$$

- Allowing x to vary in $[0,1]^n$ we get the *multilinear extension (MLE)* or *Owen extension*:

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- In terms of interpolation, the multilinear extension is the classical multilinear interpolation method, while the Lovász extension is the parsimonious (piecewise) linear interpolation

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- Sufficient conditions for the existence of difference measurement are known (Krantz et al. 1971)

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Theorem

(Dyer and Sarin 1979 + Keeney and Raiffa 1976) Suppose that the conditions for difference measurement are fulfilled and that the set of attributes is bounded. Then $\succcurlyeq^*$ satisfies weak difference independence iff $\exists$ a unique capacity μ on N and utility functions $u_1, \ldots, u_n$ s.t. F is the Owen extension (MLE) of μ.

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Theorem

(Murofushi and Sugeno 1992, 2000) Suppose $F = CI$ and $\bigcap_{i \in N} u_i(X_i)$ contains a nontrivial real interval.

1. *Suppose there are exactly two essential attributes i, j. T.f.a.e.:*
   - *Attributes i and j are preferentially independent*
   - *$\{i\}, \{j\}$ are both positive*
   - $\mu(\{i, j\}) > \max\{\mu(\{i\}), \mu(\{j\})\}$

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2. *Suppose that there at least 3 essential attributes. T.f.a.e.:*
   - *The attributes are mutually preferentially independent*
   - *μ is additive*

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- The GAI model generalizes the additive utility model ($\mathcal{S}$ is the set of singletons).

The GAI model

- The *GAI (Generalized Additive Independence)* model (Fishburn 1967) has the following form:

$$U(x) = \sum_{S \in \mathcal{S}} u_S(x_s)$$

where $\mathcal{S} \subseteq 2^N$ is a collection of nonempty subsets of N, and u_S is a utility function defined on X_S.
- The GAI model generalizes the additive utility model ($\mathcal{S}$ is the set of singletons).
- The GAI model need not satisfy weak independence

The GAI model: decomposition, *p*-additivity

- Given a GAI model U, there is no unique way to write its expression (called *decomposition*). Ex:

$$U(x) = 2x_1 + x_2 - \min(x_1, x_2) = x_1 + \max(x_1, x_2)$$

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- A GAI model U is said to be *p-additive* if there exists a decomposition

$$U(x) = \sum_{S \in \mathcal{S}} u_S(x_S)$$

such that $|S| \leq p$ for every $S \in \mathcal{S}$, with equality for some S, and no decomposition exists with all terms involving less than p variables.

Discrete GAI models and multichoice games

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- It can be shown that p-additive GAI discrete models are exactly p-additive multichoice games (in the sense of their Möbius transform)(G. and Labreuche 2016).

Outline

1. Multiattribute utility theory (MAUT)
2. The Choquet integral and MLE models
3. GAI models
4. **Interaction between criteria**

Interaction modelled by a capacity

Take two criteria $i, j \in N$.

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- Case of equality: the added value by both criteria is exactly the sum of the individual added values (*independence between criteria*)

Interaction

- (Murofushi and Soneda 1993; Owen 1972) The *interaction index* $I_{ij}(v)$ is defined as

$$I_{ij}(v) = \sum_{S \subseteq N \setminus \{i,j\}} \frac{|S|!(n-|S|-2)!}{(n-1)!}(v(S \cup \{i,j\}) - v(S \cup i) - v(S \cup j) + v(S))$$

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- The interaction index can be generalized to any set of criteria (G., 1997):

$$I_T(v) = \sum_{S \subseteq N \setminus T} \Big(\frac{|S|!(n-|S|-|T|)!}{(n-|T|+1)!} \sum_{K \subseteq T} (-1)^{|T \setminus K|} v(S \cup K) \Big).$$

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- $\{I_T(v)\}_{T \subseteq N}$ is equivalent to $\{v(S)\}_{S \subseteq N}$.

Solution of the student example

	Mathematics	Physics	Language skills
Student A	40	90	60
Student B	40	60	90
Student C	80	90	60
Student D	80	60	90

Preference is $A \succ B$ and $D \succ C$

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$v(A)$	0.3	0.3	0.2	0.4	0.7	0.7	1

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This yields

$$U(A) = 63, \quad U(B) = 60, \quad U(C) = 71, \quad U(D) = 76$$

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- The interaction index defines a transform too:

$$I^v(S) = \sum_{T \subseteq N \setminus S} \Big(\frac{t!(n-s-t)!}{(n-t+1)!} \sum_{K \subseteq S} (-1)^{|S \setminus K|} v(T \cup K) \Big)$$

Transforms of set functions

Its inverse is given by

$$v(S) = \sum_{T \subseteq N} \beta^t_{|S \cap T|} I^v(T)$$

with $\beta^l_k = \sum_{j=0}^k \binom{k}{j} B_{l-j}$ $(k \leq l)$, and $B_0, B_1, \ldots$ are the Bernoulli numbers.

$k \setminus l$	0	1	2	3	4
0	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
1		$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{30}$
2			$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{2}{15}$
3				0	$-\frac{1}{30}$
4					$-\frac{1}{30}$

Transforms of set functions

Two other important tranforms:

- The Banzhaf interaction transform:
$$I^v_{\mathrm{B}}(S) = \Big(\frac{1}{2}\Big)^{n-s}\sum_{T\subseteq N}(-1)^{|S\setminus T|}v(T)$$
and its inverse:
$$v(S) = \sum_{T\subseteq N}\frac{(-1)^{|T\setminus S|}}{2^t}I^v_{\mathrm{B}}(T)$$
- The Fourier transform:
$$\widehat{v}(S) = \frac{1}{2^n}\sum_{T\subseteq N}(-1)^{|S\cap T|}v(T)$$
and its inverse
$$v(S) = \sum_{T\subseteq N}(-1)^{|S\cap T|}\widehat{v}(T)$$
- Relation between the Banzhaf interaction and Fourier transforms:
$$\widehat{v}(S) = \Big(-\frac{1}{2}\Big)^{s}I^v_{\mathrm{B}}(S)$$

Interaction and Mutual Preferential Independence

- Fact:

$$v \text{ additive } \Leftrightarrow m^v(S) = I^v(S) = I^v_{\mathrm{B}}(S) = 0, \quad \forall S, |S| > 1$$

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  The attributes are mutually preferentially independent iff all interaction indices are null.

Interaction indices for aggregation functions

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$$\begin{aligned}\Delta_{ij} F(x) &= \Delta_i(\Delta_j F(x)) = \Delta_j(\Delta_i(x)) \\ &\quad F(b_i b_j x_{-ij}) - F(b_i a_j x_{-ij}) - F(b_j a_i x_{-ij}) + F(a_i a_j x_{-ij})\end{aligned}$$

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- Examples (with $[a, b] = [0, 1]$):

$$\Delta_{ij} \min(x) = \bigwedge_{k \neq i,j} x_k \geq 0$$

$$\Delta_{ij} \max(x) = -1 + \bigvee_{k \neq i,j} x_k \leq 0$$

$$\Delta_{ij}\Big(\frac{1}{n}\sum_i x_i\Big) = 0.$$

Interaction indices for aggregation functions

- Generalization: the *total variation of F w.r.t. $K \subseteq N$* is the function

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- The *interaction index of $K \subseteq N$ on F* is defined as the average corresponding total variation:

$$I_K(F) = \frac{1}{(b-a)^n} \int_{[a,b]^n} \frac{\Delta_K F(x)}{b-a} \, \mathrm{d}x$$

Theorem

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Note that

$$\Delta_K f^{\mathrm{Ow}}(x) = \frac{\partial^k f^{\mathrm{Ow}}}{\partial x_{|K}}(x).$$

The statistical approach: the Sobol indices

- The *Sobol indices* come from the decomposition of the variance of a multivariate function with uniform i.i.d. random variables.

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- Let $Y = f(Z)$ with $Z = (Z_1, \dots, Z_n)$ be such a multivariate function. It can be decomposed in the following way:

$$f(Z) = f_0 + \sum_{i=1}^{n} f_i(Z_i) + \sum_{i<j} f_{ij}(Z_i, Z_j) + \cdots + f_N(Z)$$

with

$$\begin{aligned} f_0 &= E(Y) \\ f_i(Z_i) &= E(Y \mid Z_i) - f_0 \\ f_{ij}(Z_i, Z_j) &= E(Y \mid Z_i, Z_j) - E(Y \mid Z_i) - E(Y \mid Z_j) + f_0 \\ &\mathit{etc}. \end{aligned}$$

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- Property: all terms in the decomposition except f_0 have zero mean

The statistical approach: the Sobol indices

It follows that the variance of Y can be decomposed as

$$\sigma^2_Y = \sum_{i=1}^{n} \sigma^2_{f_i} + \sum_{i<j} \sigma^2_{f_{ij}} + \cdots + \sigma^2_{f_N}$$

with $\sigma^2_{f_i} = E((f_i(Z_i))^2)$, etc.

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with $\sigma_{f_i}^2 = E((f_i(Z_i))^2)$, etc.
It can be shown (G. and Labreuche 2016) that if f is the Owen extension f^{Ow}, then

$$\sigma_{f_S^{\mathrm{Ow}}}^2 = \frac{1}{3^s}(\widehat{\mu}(S))^2$$

where $\widehat{\mu}$ is the Fourier transform of the capacity μ underlying f^{Ow}

Question: *Considering a GAI model U with decomposition $U(x) = \sum_{S\in\mathcal{S}} u_S(x_S)$, can we conclude that, due to the presence of the term u_S, the variables x_S are interacting?*

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We limit our discussion to the case of 2-additive GAI models.

Interaction in the GAI model

- Attributes i and j are *2-independent* if for every $x_i, y_i \in X_i, x_j, y_j \in X_j, z_{-ij} \in X_{-ij}$,

$$((x_i, x_j, z_{-ij}), (y_i, x_j, z_{-ij})) \sim^* ((x_i, y_j, z_{-ij}), (y_i, y_j, z_{-ij})), \quad (1)$$

where $\sim^*$ is the symmetric part of a quartenary relation $\succcurlyeq^*$.

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  where $\sim^*$ is the symmetric part of a quartenary relation $\succcurlyeq^*$.
- Assuming that the usual conditions of difference measurement are satisfied and that U represents $\succcurlyeq^*$, (1) translates into

$$U(x_i, x_j, z_{-ij}) + U(y_i, y_j, z_{-ij}) = U(x_i, y_j, z_{-ij}) + U(y_i, x_j, z_{-ij}).$$

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- The term u_{ij} is *trivial* if it can be put under the form
$$u_{ij}(x_i, x_j) = v_i(x_i) + v_j(x_j) + c$$

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- A decomposition is *parsimonious* if it has no trivial term.

Interaction in the GAI model

Theorem

Let U be a 2-additive GAI model and i, j be distinct attributes. Assume that the usual conditions of difference measurement are satisfied. T.f.a.e.:

1. *i, j are 2-independent for U;*
2. *There exists a parsimonious decomposition of U without a term u_{ij};*
3. *No parsimonious decomposition of U contains a term u_{ij}.*

As shown above, discrete GAI models are equivalent to multichoice games. Interactions indices have been defined for multichoice games, as well as for more general games (games on lattices):

1. M. Grabisch and Ch. Labreuche, Derivative of functions over lattices as a basis for the notion of interaction between attributes. *Annals of Mathematics and Artificial Intelligence*, Vol. 49, 2007, 151-170.
2. M. Grabisch and F. Lange, Games on lattices, multichoice games and the Shapley value: a new approach. *Mathematical Methods of Operations Research*, Vol. 65, 2007, 153-167.
3. F. Lange and M. Grabisch, The interaction transform for functions on lattices. *Discrete Mathematics*, Vol 309 (2009), 4037-4048.

Summary

In any of the approaches, the interaction for 2 criteria i, j is always of the form:

$$\mathrm{average}\big(f(\ldots,\Delta x,\Delta y,\ldots)-f(\ldots,0,\Delta y,\ldots)-f(\ldots,\Delta x,0,\ldots)\\ +f(\ldots,0,0,\ldots)\big)$$